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Low energy electron beam relaxation in gases in uniform electric fields

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Abstract. The problem of low energy electron beam relaxation in a gas between two parallel plates is extended to include accelerating or retarding electric fields. The problem is solved by two different methods, assuming that the elastic scattering is isotropic and neglecting the energy losses in elastic collisions. The first method is based on solution of the equations of moments of the electron velocity distribution function. The second uses a novel Monte-Carlo simulation scheme, which allows us to find various moments of the distribution function without the simplifying assumptions of the first method. The calculated current to the plates and the calculated density distribution between the plates obtained by the two methods are in good agreement.

1. Introduction

Low energy electron beam relaxation in a gas has been investigated by many authors (see e.g. Abramovitz and Stegun 1968, Bartels and Noak 1930, Chandrasekhar 1950, Chantry *et al* 1966). The problem was usually defined as follows. Consider the region between two parallel infinite plates in a gas. Assume that the region is filled with a gas and that one of the plates is a source of monoenergetic electrons. The energy of the electrons is assumed to be insufficient to excite or ionise the gas atoms, so that only elastic collisions are important. Neglecting the electron energy losses in elastic collisions and assuming that the space charge field of the electrons is negligible one can define the electron velocity distribution function $f(x, \theta)$, where x is the distance from the emitting plate (the cathode) and θ is the angle between the electron velocity at the point x and the x axis. If the function $f(x, \theta)$ is known, the electron density $n(x)$ can be calculated as well as the current to the plates. The integral equation for $f(x, \theta)$ is known as the equation of transfer. This equation was solved numerically by Bartels and Noak (1930). The two-stream approximation, which is frequently used in astrophysics (Chandrasekhar 1950), was also applied by Gavallas and Kagan (1965) and Chantry *et al* (1966) in order to obtain an analytical solution for $n(x)$.

In this paper a more general case, in which an accelerating or retarding uniform electric field is present between the plates, is considered. In this case one cannot use the equation of transfer and must proceed from the Boltzmann kinetic equation. In § 2 equations for the moments of the distribution function will be derived. In § 3 we use the two-stream approximation to solve the equations. A similar method was used previously by Kagan and Perel (1966) in the theory of spherical probes at intermediate pressures.

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In our calculation we neglect electron energy losses in elastic collisions. The opposite limiting case, when these losses are so important that the electron moves with an equilibrium drift velocity, was investigated by Thomson (1928).

Numerical examples will be presented in § 6, where the analytic results are compared with those obtained using a Monte-Carlo simulation method (Friedland and Fruchtman 1978). The mathematical basis of the method and the simulation procedure will be discussed in §§ 4 and 5.

2. Equations of moments

Let us neglect the electron energy losses in elastic collisions. The magnitude of the electron velocity at a given point x between the plates will be only a function of x:

$$v = \left[v_0^2 + \frac{2e}{m}\varphi(x)\right]^{1/2}. \tag{2.1}$$

Here v_0 is the initial velocity of the electrons at the cathode and $\varphi(x)$ is the potential of the electric field. Let $f(x, \theta)$ be the number of those electrons in the interval $[x, x + dx]$, having angles between their velocities and the x axis confined in the interval $[\theta, \theta + d\theta]$. Adopting the x, θ coordinate system, one can write the Boltzmann equation for the velocity distribution function f in the following form

$$\frac{\partial}{\partial x}\left(\frac{dx}{dt}f\right) + \frac{\partial}{\partial \theta}\left(\frac{d\theta}{dt}f\right) = \left(\frac{\partial f}{\partial t}\right)_c. \tag{2.2}$$

In equation (2.2)

$$\frac{dx}{dt} = v\cos\theta \qquad \frac{d\theta}{dt} = -\frac{e}{m}\frac{\sin\theta}{v}\frac{d\varphi}{dx} \tag{2.3}$$

$$\frac{\partial}{\partial x}\left(\frac{dx}{dt}f\right) = \frac{e}{m}\frac{1}{v}\frac{d\varphi}{dx}\cos\theta f + v\cos\theta\frac{\partial f}{\partial x} \tag{2.4}$$

$$\frac{\partial}{\partial \theta}\left(\frac{d\theta}{dx}f\right) = -\frac{e}{m}\frac{d\varphi}{dx}\frac{\cos\theta}{v}f - \frac{e}{m}\frac{d\varphi}{dx}\frac{\sin\theta}{v}\frac{\partial f}{\partial x}. \tag{2.5}$$

Assuming isotropic scattering, the collision term can be expressed as

$$\left(\frac{\partial f}{\partial t}\right)_c = -\nu f + \frac{\sin\theta}{2}\nu\int_0^\pi f(x, \theta')\,d\theta' \tag{2.6}$$

where $\nu(x)$ is the collision frequency. Substituting equations (2.4)–(2.6) into equation (2.3), we get the following final form of the Boltzmann equation

$$\cos\theta\frac{\partial f}{\partial x} - g(x)\sin\theta\frac{\partial f}{\partial \theta} = -\frac{1}{\lambda}f + \frac{\sin\theta}{2\lambda}\int_0^\pi f(x, \theta')\,d\theta' \tag{2.7}$$

where $\lambda = v/\nu$ is the electron mean free path and

$$g(x) = \frac{e}{m}\frac{1}{v^2}\frac{d\varphi}{dx}. \tag{2.8}$$

Let us now define the following moments of the distribution function

$$n(x) = \int_0^\pi f\,d\theta \qquad H(x) = \int_0^\pi f\cos\theta\,d\theta \qquad K(x) = \int_0^\pi f\cos^2\theta\,d\theta. \tag{2.9}$$

The function $n(x)$ is the electron density and $H(x)$ gives the total electron current density $J(x)$:

$$J(x) = v(x)\, H(x). \tag{2.10}$$

Integrating equation (2.7) with respect to v, one has

$$\frac{dH}{dx} + gH = 0 \tag{2.11}$$

or

$$Hv = J_0 = \text{constant}. \tag{2.12}$$

On multiplying equation (2.7) by $\cos\theta$ and integrating it with respect to v, one gets the second equation for the moments

$$\frac{dK}{dx} + 2gK - gn = -\frac{1}{\lambda} H. \tag{2.13}$$

3. Approximate solution

In order to find the moments of the distribution function we now assume that the distribution function $f(x, \theta)$ has the following form

$$f(x, \theta) = f_0(x)\, \delta(\theta) + \begin{cases} f_1(x) \sin\theta & 0 \leqslant \theta \leqslant \pi/2 \\ f_2(x) \sin\theta & \pi/2 \leqslant \theta \leqslant \pi. \end{cases} \tag{3.1}$$

Here

$$f_0 = \frac{j_0}{v} \exp\left[- \int_0^x \frac{dx}{\lambda} \right]$$

where j_0 is the initial current density of the electron beam, $\delta(\theta)$ is the Dirac delta function and f_1 and f_2 depend only on x. The term $f_0(x)\, \delta(\theta)$ corresponds to the distribution of the unscattered electrons at the point x. The second term in equation (3.1) describes the electrons scattered at least once during their motion between the plates. We assume that the whole region $0 < \theta < \pi$ can be divided into two parts: $0 < \theta < \pi/2$ (electrons moving in the forward direction) and $\pi/2 < \theta < \pi$ (electrons moving in the backward direction). In each of these angles the electron distribution is assumed to be isotropic in space with different functions $f_1(x)$ and $f_2(x)$, i.e. with different numbers of electrons moving in opposite directions.

Although equation (3.1) is not a solution of the kinetic equation (2.7), the functions f_1 and f_2 can be found so that equation (3.1) would satisfy the equations for the moments. This approach is justified by the fact that one is usually interested in the moments of the distribution function (such as the electron density and the currents to the plates) rather than in the detailed form of the distribution function itself.

On using equation (3.1), the moments n, H and K can be expressed as

$$n = f_0 + f_1 + f_2 \qquad H = f_0 + \tfrac{1}{2}(f_1 - f_2) \qquad K = f_0 + \tfrac{1}{3}(f_1 + f_2). \tag{3.2}$$

Let us now define two additional functions

$$n' = f_1 + f_2 \qquad S = f_1 - f_2. \tag{3.3}$$

Then

$$f_1 = \tfrac{1}{2}(n' + S) \qquad f_2 = \tfrac{1}{2}(n' - S) \tag{3.4}$$

and

$$n = f_0 + n' \qquad H = f_0 + \tfrac{1}{2}S \qquad K = f_0 + \tfrac{1}{3}n'. \tag{3.5}$$

Substituting these expressions into equations (2.12) and (2.13) for the moments one has

$$S = \frac{2J_0}{v} - 2f_0 \tag{3.6}$$

$$\frac{dn'}{dx} - gn' = -\frac{3J_0}{\lambda v} + \frac{3f_0}{\lambda}. \tag{3.7}$$

Let λ = constant for simplicity. Then equation (3.7) has a solution

$$n' = v(x)\left(C_0 - \frac{3J_0}{\lambda}I_1 + \frac{3f_0}{\lambda}I_2\right) \tag{3.8}$$

where

$$I_1 = \int_0^x \frac{dx}{v^2} \qquad I_2 = \int_0^x \frac{1}{v^2}e^{-x/\lambda}\,dx \tag{3.9}$$

and C_0 is a constant of integration. Finally according to equation (3.4)

$$f_1 = \frac{1}{2}C_0 v - \frac{3}{2}\frac{J_0}{\lambda}vI_1 + \frac{3}{2}\frac{j_0}{\lambda}vI_2 + \frac{J_0}{v} - f_0 \tag{3.10}$$

$$f_2 = \frac{1}{2}C_0 v - \frac{3}{2}\frac{J_0}{\lambda}vI_1 + \frac{3}{2}\frac{j_0}{\lambda}vI_2 - \frac{J_0}{v} + f_0. \tag{3.11}$$

In order to find C_0 and J_0 one can use the boundary conditions

$$f_1(0) = 0 \qquad f_2(d) = 0 \tag{3.12}$$

where d is the distance between the plates. These boundary conditions correspond to the case of perfectly absorbing plates. (Note that the current to the anode in this case is equal to J_0.) Then

$$J_0 = j_0\left[\left(e^{-d/\lambda} + \frac{v_d^2}{v_0^2} + \frac{3v_d^2}{2\lambda}I_2(d)\right)\left(1 + \frac{v_d^2}{v_0^2} + \frac{3}{2}\frac{v_d^2}{\lambda}I_1(d)\right)^{-1}\right] \tag{3.13}$$

$$C_0 = \frac{2j_0}{v_0^2}\left[\left(1 - e^{-d/\lambda} + \frac{3v_d^2}{2\lambda}[I_1(d) - I_2(d)]\right)\left(1 + \frac{v_d^2}{v_0^2} + \frac{3v_d^2}{2\lambda}I_1(d)\right)^{-1}\right]. \tag{3.14}$$

4. Relation between the moments of the distribution function and the parameter of the random walk of a single electron

We now describe the mathematical basis for the Monte-Carlo simulation scheme, which allows one to find various moments of the distribution function directly without the simplifying assumptions used in the previous section. We will consider a more general problem and will place N additional, partially absorbing parallel infinite grids between the cathode and the anode. All the grids may be at different electrostatic potentials. Then there exist the following two theorems.

4.1. Theorem 1

Let j_0 be the current density of the electrons emitted at the cathode and let P_m be the probability that a single electron emitted at the cathode will be collected on the mth grid. Then the current density, flowing to this grid

$$j_m = j_0 P_m. \tag{4.1}$$

Proof: let $P_m(t)\,\mathrm{d}t$ be the probability that the electron will be collected on the mth grid during the time interval $\mathrm{d}t$. Then the current density to the grid is

$$j_m = \int_0^\infty j_0(t-\tau)\,P_m(\tau)\,\mathrm{d}\tau. \tag{4.2}$$

But $j_0(t-\tau) = j_0$ and, by definition, $P_m = \int_0^\tau P_m(\tau)\,\mathrm{d}\tau$. This proves equation (4.1). The currents to the cathode or to the anode also can be found from (4.1).

We now use Theorem 1 in the proof of the second theorem, which forms the basis for the Monte-Carlo calculation of the electron density distribution between the plates.

4.2. Theorem 2

Again let j_0 be the current density of the electrons emitted at the cathode. Then the electron density at a given point x is given by

$$n(x) = \lim_{M_0\to\infty}\left[\frac{j_0}{M_0}\sum_{l=1}^{M_0}\left(\sum_{\substack{\text{no. of}\\ \text{crossings}}}\frac{1}{|v_{l\|}|}\right)\right] \tag{4.3}$$

where the parameters refer to the following gedanken experiment. Consider M_0 electrons emitted from the cathode and following one by one in the gap between the plates. Each time one of these electrons (say the lth electron) crosses an imaginary plane parallel to the cathode and located at a given point x, the term $1/|v_{l\|}|$, where $v_{l\|}$ is parallel to the x axis component of the electron velocity at the crossing moment, is added to the sum in equation (4.3). The electron density $n(x)$ in our real experiment is then obtained by summing over all the electrons in the gedanken experiment and taking the limit in equation (4.3) as the number of these electrons (M_0) goes to infinity.

Let us introduce a virtual grid, characterised by an infinitesimal absorption coefficient K, at the position of the plane at the point x. The current density collected by this grid as a result of the electrons crossing the grid at an angle θ is

$$j_\theta = K n_\theta(x)\,|v_{\theta\|}(x)| \tag{4.4}$$

where $n_\theta(x)$ is the density of the electrons passing the grid at angle θ, and $v_{\theta\|} = v(x)\cos\theta$ is the velocity component along the x axis. On the other hand, according to Theorem 1

$$j_\theta = j_0 P_\theta = j_0 K \lim_{M_0\to\infty}\left(\frac{1}{M_0}\sum_{l=1}^{M_0} M_{l\theta}\right). \tag{4.5}$$

P_θ is the probability of an electron, emitted from the cathode, being absorbed at the grid, when it passes the grid at an angle θ. The term in the brackets is the average number of times that an electron crosses the grid at this angle.

On comparing equations (4.4) and (4.5), one has

$$n_\theta(x) = \lim_{M_0 \to \infty} \left[\frac{j_0}{M_0} \sum_{l=1}^{M_0} \frac{M_{l\theta}}{|v_{0\|}|} \right] \tag{4.6}$$

and therefore the electron density is given by

$$n(x) = \sum_\theta n_\theta(x) = \lim_{M_0 \to \infty} \left[\frac{j_0}{M_0} \sum_{l,\theta} \frac{M_{l\theta}}{|v_{\theta\|}|} \right] = \lim_{M_0 \to \infty} \left[\frac{j_0}{M_0} \sum_{l=1}^{M_0} \left(\sum_{\substack{\text{no. of} \\ \text{crossings}}} \frac{1}{|v_{l\|}|} \right) \right]. \tag{4.7}$$

This completes the proof of the theorem.

The power of this theorem is that it allows one to find the electron density distribution in our real system by simulating the random walk of M_0 electrons, and just adding the terms of the form $1/|v_\||$ to the sum in equation (4.3), each time an electron passes the point x. By choosing a reasonable number M_0 in such a gedanken experiment, the quantity in the square brackets in (4.3) can give a good approximation for $n(x)$.

5. The simulation method

The random walk of the electrons between the electrodes was simulated in a fashion similar to that used by Friedland (1977). In addition, we adopted the null-event method of Lin and Bardsley (1978) which simplifies the simulation of collisions in cases when the collision frequency ν is not constant as, for example, in our case in the presence of the electric field.

The simulation algorithm is as follows. We consider a test electron that starts its motion at the cathode at a time t_0, having initial velocity v_0 in the direction of the x axis. We first simulate the time of flight of the electron till the first collision with an atom. Since the collision frequency $\nu = v(x)/\lambda$ in our problem depends on position, it is convenient to introduce a new constant collision frequency $\nu_{\text{eff}} = v_{\max}/\lambda$, where $v_{\max}^2 = v_0^2 + 2eEd/m$ is the maximal possible velocity of the electrons in the gap. We assume that there exist two types of collisions: (i) the real collisions with collision frequency ν after which the electron is scattered at a certain angle θ; (ii) null-collisions, characterised by collision frequency $\nu_{\text{null}} = \nu_{\text{eff}} - \nu$, which do not influence the electron motion. Since now the total collision frequency ν_{eff} is constant, the time interval Δt that passes until the first collision is given by Friedland (1977)

$$t = -\ln \gamma / \nu_{\text{eff}} \tag{5.1}$$

where γ is a random number from a sequence of computer generated random numbers with a uniform distribution in the interval (0, 1).

When Δt is known the type of the collision is identified by generating a new random number γ and if $\gamma \leqslant \nu_{\text{null}}/\nu_{\text{eff}}$ it is decided that a null collision takes place. The frequency ν_{null} in this simulation is evaluated at the point the electron would occur at after a time interval Δt. Then the new free walk is simulated by assuming that the electron continues its motion with the same velocity and direction as before the collision. The new interval Δt is added to the previous one, the new type of collision is simulated and so on till one of the random numbers γ, during the simulation of a collision type, satisfies $\gamma > \nu_{\text{null}}/\nu_{\text{eff}} = 1 - \nu/\nu_{\text{eff}}$. Then it is decided that a real collision takes place.

Once a real collision has occurred, the scattering angle is simulated. For simplicity we assume that the scattering is isotropic so that the values of $S = \cos\theta$ (θ is the scattering angle) are uniformly distributed in the interval $(-1, +1)$. Therefore, the simulation formula for S is given by

$$S = 2\gamma - 1. \tag{5.2}$$

At this stage of the computations there is enough data to proceed to the next collision. The random walk of the electron is continued in a similar fashion until the electron reaches one of the electrodes and is absorbed. The simulation process is repeated with M_0 test electrons. During the process each time an electron crosses a plane, passing through a point x between the electrodes, a new term $1/v_\parallel$ is added to the sum in equation (4.3), which allows one to find the electron density $n(x)$ if the number M_0 of the test electrons is large enough for reliable statistics.

6. Numerical examples

As a first example let us consider the case of zero field between the plates. In this case

$$v_d = v_0 \qquad I_1 = \frac{x}{v_0^2} \qquad I_2 = \frac{\lambda}{v_0^2}\left[1 - \exp\left(-\frac{x}{\lambda}\right)\right]$$

and from equations (3.13) and (3.14) one gets

$$J_0 = j_0 \frac{5 - \mathrm{e}^{-d/\lambda}}{4 + 3(d/\lambda)} \tag{6.1}$$

$$C_0 = \frac{2j_0}{v_0^2} \frac{\mathrm{e}^{-d/\lambda} + (3\mathrm{d}/\lambda) - 1}{4 + (3\mathrm{d}/\lambda)}. \tag{6.2}$$

According to equations (3.5) and (3.8) the normalised electron density is

$$\rho(x) = \frac{n(x)}{(j_0/v_0)} = \mathrm{e}^{-x/\lambda} + \frac{C_0 v_0^2}{j_0} + 3(1 - \mathrm{e}^{-x/\lambda}) - \frac{3J_0}{j_0}\frac{x}{\lambda}. \tag{6.3}$$

The dependence of $\rho(x)$ on x/d is given in figure 1 for various values of d/λ (the full curves). The dashed curves show the numerical results of Bartels and Noak (1930). It can be seen in the figure that for $d/\lambda \gg 1$ the agreement is very good. We now proceed to a more general case of a uniform electric field between the plates. In this case

$$I_1 = \frac{m}{2eE} \ln\left(1 + \frac{2eEx}{mv_0^2}\right) \tag{6.4}$$

$$I_2 = \frac{m}{2eE} \mathrm{e}^a \int_a^{(x/\lambda)+a} \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t \tag{6.5}$$

where

$$a = \frac{mv_0^2}{2eE}. \tag{6.6}$$

According to the sign of E, one now has two cases. For $E > 0$, a is positive and equation (6.5) can be written as

$$I_2 = \frac{m}{2eE} e^a \left[E_1(a) - E_1\left(a + \frac{x}{\lambda}\right) \right] \tag{6.7}$$

where

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt = -0.577 - \ln z - \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n \cdot n!}$$

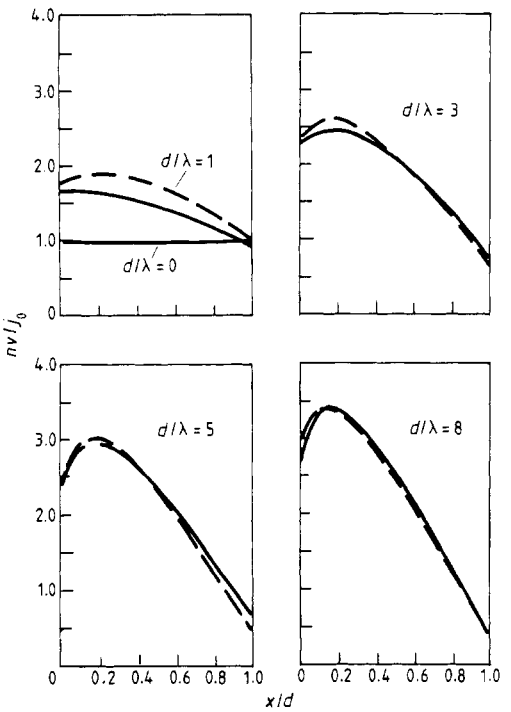

Figure 1. Normalised electron density nv/j_0 versus normalised distance x/d for various normalised gas pressures d/λ for the case of zero electric field (solid line). The dashed line represents the numerical results of Bartels and Noak (1930).

is the exponential integral (Abramovitz and Stegun 1964). When $E < 0$, we have $a < 0$ and if the initial electron energy is large enough to reach the retarding plate, then

$$\frac{x}{\lambda} + a = \frac{1}{\lambda eE}\left(\frac{mv_0^2}{2} + eEx\right) < 0.$$

Therefore both limits in the integral in equation (6.5) are negative. In this case

$$I_2 = -\frac{m}{2eE} e^a \left[E_i(-a) - E_i\left(-a - \frac{x}{\lambda}\right) \right] \tag{6.8}$$

where

$$E_i(z) = 0.577 + \ln z + \sum_{n=1}^{\infty} \frac{z^n}{n \cdot n!}.$$

Substituting equations (6.4) and (6.7) into equations (3.13) and (3.14) we find for $E > 0$

$$A_1 = \frac{J_0}{j_0}\left\{e^{-d/\lambda} + \frac{v_d^2}{v_0^2} + \frac{3}{2}\frac{v_d^2}{\lambda}\frac{m}{2eE}e^a\left[E_1(a) - E_1\left(a + \frac{d}{\lambda}\right)\right]\right\}$$
$$\times\left[1 + \frac{v_d^2}{v_0^2} + \frac{3}{2}\frac{v_d^2}{\lambda}\frac{m}{2eE}\ln\left(1 + \frac{2eEd}{mv_0^2}\right)\right]^{-1} \tag{6.9}$$

$$A_2 = \frac{v_0^2 c_0}{j_0} = \left[\left[2(1 - e^{-d/\lambda}) + \frac{3mv_d^2}{2eE\lambda}\left\{\ln\left(1 + \frac{2eEd}{mv_0^2}\right) - e^a\left[E_1(a) - E_1\left(a + \frac{d}{\lambda}\right)\right]\right\}\right]\right]$$
$$\times\left[1 + \frac{v_d^2}{v_0^2} + \frac{3}{2}\frac{mv_d^2}{2eE\lambda}\ln\left(1 + \frac{2eEd}{mv_0^2}\right)\right]^{-1}. \tag{6.10}$$

We now define the following nondimensional parameters

$$y = \frac{x}{d} \qquad z = \frac{d}{\lambda} \qquad r = \frac{2eEd}{mv_0^2}. \tag{6.11}$$

Then equations (6.9) and (6.10) can be written as

$$A_1 = \left\{e^{-z} + 1 + r + \frac{3}{2}\frac{z}{r}(1 + r)\,e^{-z/r}\left[E_1\left(\frac{z}{r}\right) - E_1\left(\frac{z}{r} + z\right)\right]\right\}$$
$$\times\left[2 + r + \frac{3}{2}\frac{z}{r}(1 + r)\ln(1 + r)\right]^{-1} \tag{6.12}$$

$$A_2 = 2\left[\left[(1 - e^{-z}) + \frac{3}{2}\frac{z}{r}(1 + r)\left\{\ln(1 + r) - e^{-z/r}\left[E_1\left(\frac{z}{r}\right) - E_1\left(\frac{z}{r} + z\right)\right]\right\}\right]\right]$$
$$\times\left[2 + r + \frac{3}{2}\frac{z}{r}(1 + r)\ln(1 + r)\right]^{-1}. \tag{6.13}$$

On substituting A_1 and A_2 into equation (3.8), one gets the following expression for normalised electron density between the plates

$$\frac{n}{(j_0/v)} = e^{-zy} + (1 + ry)$$
$$\times\left[\left[A_2 - 3\frac{z}{r}\left\{A_1\ln(1 + ry) - e^{z/r}\left[E_1\left(\frac{z}{r}\right) - E_1\left(\frac{z}{r} + zy\right)\right]\right\}\right]\right]. \tag{6.14}$$

In the case of a retarding field ($E < 0$) one has to use the function $-E_i(-z)$ instead of $E_i(z)$ in equations (6.11)–(6.14).

The dependence of nv/j_0 on y at $z = 5$ is shown in figure 2 for various values of r. The full and the dashed curves in the figure correspond to the cases of retarding ($r < 0$) and accelerating ($r > 0$) electric fields respectively. Figure 3 shows the ratio between the current density to the anode, J_0 and the initial current density j_0 of the electron beam, as a function of the parameter r for various values of z. The dependence of nv/j_0 on y at $r = 1$ obtained by using equation (6.14) (dashed curves) and by the Monte-Carlo method (squares) is shown in figure 4. One can see from this figure that the results from both methods are in good agreement.

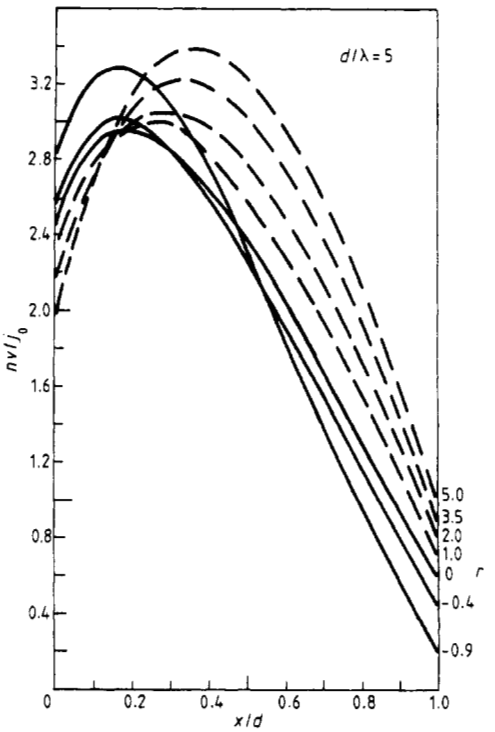

Figure 2. Normalised electron density nv/j_0 versus normalised distance x/d for various normalised electric field $r = 2eEd/mv_0^2$. All curves correspond to the case of $d/\lambda = 5$.

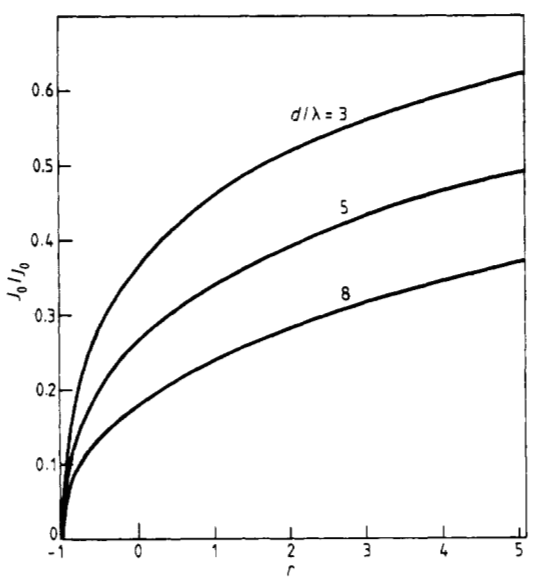

Figure 3. Normalised current to the anode J_0/j_0 versus normalised electric field r for various normalised gas pressures.

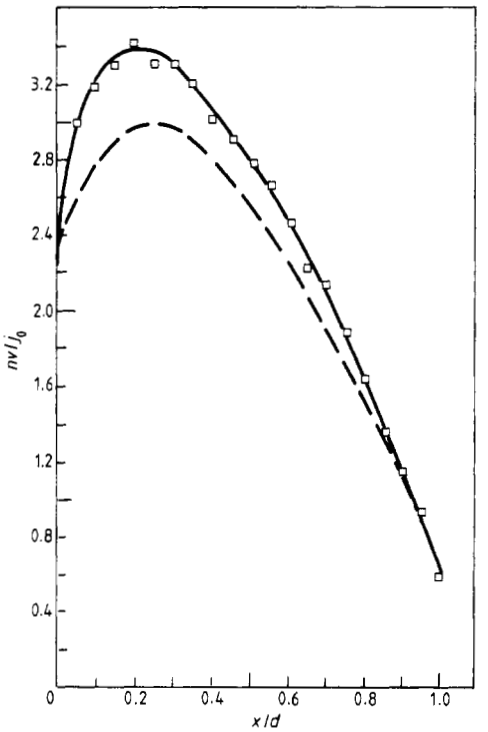

Figure 4. Normalised electron density nv/j_0 versus normalised distance for $r = 1$. The full and dashed curves correspond to the Monte-Carlo and analytical methods respectively.

7. Conclusions

We applied two different methods to the problem of low energy electron beam relaxation in a gas in the presence of a uniform electric field. The first is the analytic method, which allows one to obtain a simple formula for the electron density distribution $n(x)$ in the gap between the plates, as well as an expression for the current density J_0 to the anode. The values J_0/j_0 and $1 - J_0/j_0$ thus obtained can be interpreted as the probabilities for an electron to reach, at a certain time, the anode or to return to the cathode. Because of the strong dependence on the collision frequency, these probabilities can be used in interpreting results of electron collision experiments, where the electrons enter the collision region by passing through a small hole or a slit in the cathode. It must be mentioned that equation (3.8) for $n(x)$ can be applied even if the field $E(x)$ in the gap is influenced by the space charge and is no longer uniform. In this case the field $E(x)$ itself is also unknown. One can, however, suggest an iterative procedure for calculating both $E(x)$ and $n(x)$ corresponding to a given initial current j_0 from the cathode. As a first iteration one can assume that the field $E_0(x)$ is uniform and apply equation (3.8) to find the corresponding electron density $n_0(x)$. Then one can solve numerically Poisson's equation and get a new electric field $E_1(x)$. The new density distribution $n_1(x)$ is then calculated, again applying equation (3.8), and so on. The iteration procedure is continued until convergence is achieved. Although the analytic method is very convenient, it is limited to relatively low values of E/p and λ/d.

The second method used in the paper is based on the Monte-Carlo type simulation. This method is more general and uses very few assumptions. It can be applied for any given value of E/p and λ/d and, in fact, is more efficient for larger values of E/p and λ/d, where the analytic method is invalid, since fewer collisions are involved in these cases, thus considerably reducing the computing time. The iteration procedure, described above, can also be implemented using the density distributions obtained from computer simulations. The computer code based on the Monte-Carlo method is extremely simple and the only limitation is imposed by the increasing amount of computing time necessary to achieve good accuracy when the value of λ/d is decreasing. Thus, in conclusion, the analytic and the Monte-Carlo methods applied to the problem considered in the paper not only support, but complement each other, as more suitable for different ranges of parameters.

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